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Applications of noise theory to plasma fluctuations

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Noise theory is used to study the temporal correlations of stationary random fluctuations that are homogeneous in space. Statistical properties of the fluctuations, such as the power spectrum and the correlation function, are computed. The results are compared with the observed plasma density fluctuations from tokamak experiments.

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I. INTRODUCTION

The fluctuation-induced transport of particles and energy is important to plasma confinement in a fusion device [1]. Various methods have been developed for solving fluctuation problems [2–6]. In this paper, we use noise theory [7,8] to study the temporal correlations of stationary and Markovian fluctuations that are homogeneous and isotropic in space. The results are compared with the observed plasma density fluctuations during a steady-state discharge [9].

We define random fluctuations $\boldsymbol{\alpha}=\{\alpha_i\}$ to have zero mean value $\langle\alpha_i(t)\rangle=0$, where $\langle\cdots\rangle$ denotes the ensemble average. The statistical properties of stationary fluctuations can be quantified by the correlation function and the power spectrum.

The correlation function is defined as $\langle\alpha_i(t)\alpha_j(t-\tau)\rangle=C_{ij}(\tau)$, where τ is the time difference. If the Fourier transform of $\alpha_i(t)$ is denoted by $\alpha_i(\omega)$, then the power spectrum is defined as $\langle\alpha_i(\omega)\alpha_j^*(\omega')\rangle=S_{ij}(\omega)\delta(\omega-\omega')$, where the asterisk denotes the complex conjugate.

According to the Wiener-Khintchin theorem [7], the Fourier transform of the correlation function equals the power spectrum

$$S_{ij}(\omega)=\int_{-\infty}^{\infty}d\tau\, e^{-i\omega\tau}C_{ij}(\tau), \tag{1}$$

and the inverse Fourier transform of the power spectrum equals the correlation function

$$C_{ij}(\tau)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega\tau}S_{ij}(\omega), \tag{2}$$

where $\omega=2\pi f$. If we let $\boldsymbol{S}_+(\omega)=\int_0^\infty d\tau\, e^{-i\omega\tau}\boldsymbol{C}(\tau)$, then Eq. (1) can be written as [7]

$$\boldsymbol{S}(\omega)=\boldsymbol{S}_+ + \boldsymbol{S}_+^{T*}, \tag{3}$$

where T denotes the matrix transpose.

The conditional probability $P(\boldsymbol{\alpha},t|\boldsymbol{\alpha}_o)$ is the probability that $\boldsymbol{\alpha}(t)$ will take the value $\boldsymbol{\alpha}$ at time t given $\boldsymbol{\alpha}(t=0)=\boldsymbol{\alpha}_o$. The conditional mean is defined as [7]

$$\langle\boldsymbol{\alpha}\rangle_{\alpha o}=\int d\boldsymbol{\alpha}\, P(\boldsymbol{\alpha},t|\boldsymbol{\alpha}_o)\boldsymbol{\alpha}, \tag{4}$$

where $\langle\cdots\rangle_{\alpha o}$ denotes conditional mean. For Markovian systems, the evolution of the conditional mean is modeled by [7]

$$\frac{d}{dt}\langle\alpha_i\rangle_{\alpha o}=-\Lambda_{ij}\langle\alpha_j\rangle_{\alpha o}, \tag{5}$$

where $\boldsymbol{\Lambda}$ is the response matrix.

Time reversibility [7] requires the correlation function to be even and symmetric: $\boldsymbol{C}(\tau)=\boldsymbol{C}(-\tau)=\boldsymbol{C}^T(\tau)$. Thus the corresponding power spectrum is real. Then the correlation function takes the form [7]

$$\boldsymbol{C}(\tau)=e^{-\boldsymbol{\Lambda}|\tau|}\boldsymbol{C}(\tau=0), \tag{6}$$

where $\boldsymbol{C}(\tau=0)$ represents correlations at one instant of time. Thus [7]

$$\boldsymbol{S}_+^T=\boldsymbol{S}_+=(\boldsymbol{\Lambda}+i\omega\boldsymbol{I})^{-1}\cdot\boldsymbol{C}(\tau=0), \tag{7}$$

where $\boldsymbol{I}$ is the unit matrix. Hence Eq. (3) becomes [7]

$$\boldsymbol{S}(\omega)=2\,\mathrm{Re}\,\boldsymbol{S}_+(\omega)=2\,\mathrm{Re}\,\boldsymbol{M}^{-1}\cdot\boldsymbol{C}(\tau=0), \tag{8}$$

where $\boldsymbol{M}=\boldsymbol{\Lambda}+i\omega\boldsymbol{I}$.

To study spatial correlations, we choose a set of random variables $\alpha_i\to\alpha(\boldsymbol{x})$ so that a Markovian description of a physical system is possible [7]. The $\boldsymbol{\alpha}$ we consider represents the fluctuations of a single parameter, such as particle density, at different spatial points. Thus we replace the discrete index by the spatial coordinate [7]. For example, Eq. (5) becomes

$$\frac{\partial}{\partial t}\langle\alpha(\boldsymbol{x})\rangle_{\alpha o}+\int d^3x'\,\Lambda(\boldsymbol{x},\boldsymbol{x}')\langle\alpha(\boldsymbol{x}')\rangle_{\alpha o}=0. \tag{9}$$

For homogeneous fluctuations, the correlations between different spatial points depend only on the differences $\Delta\boldsymbol{x}=\boldsymbol{x}-\boldsymbol{x}'$. Thus the matrix products become convolution integrals [6]. For example, $\boldsymbol{M}\cdot\boldsymbol{M}^{-1}=\boldsymbol{I}$ becomes

$$\int d^3\boldsymbol{x}''M(\boldsymbol{x}-\boldsymbol{x}'')M^{-1}(\boldsymbol{x}''-\boldsymbol{x}')=\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{10}$$

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Then the spatial Fourier transform of Eq. (10) gives $M^{-1}(\boldsymbol{k})=1/M(\boldsymbol{k})$. Similarly, Eq. (8) becomes

$$S(\boldsymbol{x}-\boldsymbol{x}',\omega) = 2\,\mathrm{Re}\int d^3x'' M^{-1}(\boldsymbol{x}-\boldsymbol{x}'')C(\boldsymbol{x}''-\boldsymbol{x}',\tau=0), \tag{11}$$

where $M(\boldsymbol{x}-\boldsymbol{x}')=\Lambda(\boldsymbol{x}-\boldsymbol{x}')+i\omega\delta(\boldsymbol{x}-\boldsymbol{x}')$. Then the spatial Fourier transform of Eq. (11) gives

$$S(\boldsymbol{k},\omega) = 2\,\mathrm{Re}\,\frac{C(\boldsymbol{k},\tau=0)}{\Lambda(\boldsymbol{k})+i\omega}. \tag{12}$$

II. DIFFUSION MODEL OF RANDOM FLUCTUATIONS

For Markovian systems with diffusion response, the conditional mean of random fluctuations is modeled by the diffusion equation [7]

$$\frac{\partial}{\partial t}\langle\alpha\rangle_{\alpha o} = D\nabla^2\langle\alpha\rangle_{\alpha o}, \tag{13}$$

where D is the effective diffusion coefficient. Comparing Eq. (9) with Eq. (13), we obtain the diffusion response function

$$\Lambda(\boldsymbol{x}-\boldsymbol{x}') = -D\nabla^2\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{14}$$

Then the spatial Fourier transform of Eq. (14) gives

$$\Lambda(k) = Dk^2, \tag{15}$$

where $k=|\boldsymbol{k}|$ is the magnitude of the wave vector.

Experiments [10] show that the spatial correlation of plasma fluctuations can be approximated by an exponential decay. Hence we assume that for isotropic fluctuations, the spatial correlations at one instant of time take the form

$$\langle\alpha(\boldsymbol{x},t)\alpha(\boldsymbol{x}',t)\rangle = C(\Delta\boldsymbol{x},\tau=0) = C_o e^{-|\Delta\boldsymbol{x}|/\lambda_c}, \tag{16}$$

where $C_o=\langle\alpha^2(\boldsymbol{x},t)\rangle$ is the mean square value of the fluctuations.

Next we assume that the fluctuation energy is determined by fluctuation wavelengths that are shorter than the correlation length λ_c. Hence for isotropic fluctuations, we define $E(k)$, the wavelength spectrum of the fluctuation energy, by the condition

$$\langle\alpha^2(\boldsymbol{x},t)\rangle = \int d^3k\, C(\boldsymbol{k},\tau=0) = \int_{k_c}^{\infty} dk\, E(k), \tag{17}$$

where $k_c=2\pi/\lambda_c$ is the cutoff wave number.

The k dependence of $E(k)$ may be inferred from $C(k,\tau=0)$, the Fourier transform of Eq. (16), as follows: in three dimensions, $E(k)\propto k^2C(k)$, where $C(k)\propto\lambda_c^3/(1+\lambda_c^2k^2)^2 \approx 1/(\lambda_c k^4)$ for $\lambda_c k>2\pi$; in two dimensions, $E(k)\propto kC(k)$, where $C(k)\propto\lambda_c^2/(1+\lambda_c^2k^2)^{3/2}\approx 1/(\lambda_c k^3)$ for $\lambda_c k>2\pi$; in one dimension, $E(k)\propto C(k)$, where $C(k)\propto\lambda_c/(1+\lambda_c^2k^2) \approx 1/(\lambda_c k^2)$ for $\lambda_c k>2\pi$.

In summary, under the assumption of fluctuation wave numbers $k>k_c$, regardless of the spatial dimension, $E(k) \propto k^{-2}$, then the proportionality constant is obtained from the condition Eq. (17). Thus

$$E(k) = C_o\frac{k_c}{k^2}. \tag{18}$$

Note that the autopower spectral density function is given by

$$S(\Delta\boldsymbol{x}=0,\omega) = \int d^3k\, S(\boldsymbol{k},\omega). \tag{19}$$

For isotropic fluctuations, using Eqs. (12) and (17), we write the autopower spectrum Eq. (19) as

$$S(\omega) = \int_{k_c}^{\infty} dk\, E(k)\frac{2Dk^2}{(Dk^2)^2+\omega^2} = \int_{k_c}^{\infty} dk \frac{2C_o k_c D}{D^2k^4+\omega^2} \tag{20}$$

where Eqs. (15) and (18) are used. Note that the autopower spectrum Eq. (20), which indicates the frequency distribution of the fluctuation energy, satisfies the condition that the area under the spectrum is equal to the mean square value of the fluctuations,

$$\langle\alpha^2(\boldsymbol{x},t)\rangle = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}S(\Delta\boldsymbol{x}=0,\omega). \tag{21}$$

We can write the diffusion coefficient of random fluctuations as

$$D = f_c/k_c^2 = f_c\lambda_c^2/(2\pi)^2, \tag{22}$$

where f_c has the unit of frequency.

If we let $y=k\sqrt{\frac{D}{\omega}}$, then Eq. (20) becomes

$$S(\omega) = 2C_o\frac{\sqrt{f_c}}{\omega^{3/2}}\int_{\sqrt{f_c/\omega}}^{\infty}\frac{dy}{y^4+1} \approx \frac{2C_o}{3f_c}H(-\omega^2/f_c^2) \tag{23}$$

where $H(z)$ represents the hypergeometric function ${}_2F_1(3/4,1;7/4;z)$. Thus f_c turns out to be the spectrum width. Note that even if the maximum wave number k_{max} of the measurement is finite, for $k_{max}^3\gg k_c^3$, the result is still a good approximation.

Then the inverse Fourier transform of Eq. (23) gives the autocorrelation function, which quantifies the correlations between different times at one spatial point,

$$C(\Delta\boldsymbol{x}=0,\tau)/C_o = e^{-|\tau|f_c} - \sqrt{\pi|\tau|f_c}\,\mathrm{erfc}(\sqrt{|\tau|f_c}) \tag{24}$$

where $\mathrm{erfc}(z)$ is the complementary error function. Thus f_c is also the decay rate, and $1/f_c=\tau_c$ is the correlation time.

In summary, by using the pure diffusion model, we have obtained the analytic functions of the autopower spectrum and the autocorrelation. In the pure diffusion model, the spectrum has a single peak at zero frequency, and the autocorrelation purely decays away. We find that the diffusion gives rise to the decay feature of the autocorrelation and a broad frequency spectrum of random fluctuations.

Finally, to compare with the experimental data, the decay and oscillation features of the autocorrelation are simply modeled by the diffusion decay part Eq. (24) times a cosine with a constant oscillation frequency. That is,

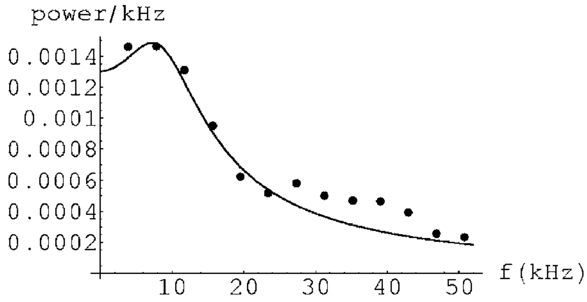

FIG. 1. The dots represent the autopower spectrum data of the fractional plasma density fluctuations from tokamak experiments [9]. The curve is given by Eq. (26).

$$C(\tau)/C_o = \cos(\omega_o \tau)[e^{-|\tau|f_c} - \sqrt{\pi|\tau|f_c}\,\mathrm{erfc}(\sqrt{|\tau|f_c})] \quad (25)$$

where $\omega_o/(2\pi)=f_o$ is the oscillation frequency. It follows from Eqs. (23) and (25) that the autopower spectrum peak is shifted away from zero frequency. That is,

$$2S(f) = 2\frac{1}{2}\left(\frac{2C_o}{3f_c}\right)\{H[-(\omega+\omega_o)^2/f_c^2] + H[-(\omega-\omega_o)^2/f_c^2]\}. \quad (26)$$

Thus f_o is also the peak frequency of the spectrum. It follows from Eq. (21) that the fluctuation power can then be expressed as

$$\langle\alpha^2\rangle = \int_0^\infty df\, 2S(f). \quad (27)$$

To explain the physical meaning of the constant oscillation frequency, the relaxation of the conditional mean is modeled by the convection-diffusion equation [7]

$$\frac{\partial}{\partial t}\langle\alpha\rangle_{\alpha o} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\langle\alpha\rangle_{\alpha o} - D\nabla^2\langle\alpha\rangle_{\alpha o} = 0, \quad (28)$$

where $\boldsymbol{U}$ is the flow velocity. Thus the response function becomes

$$\Lambda(\boldsymbol{x}-\boldsymbol{x}') = -D\nabla^2\delta(\boldsymbol{x}-\boldsymbol{x}') - \boldsymbol{U}\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{x}'). \quad (29)$$

The spatial Fourier transform of Eq. (29) gives

$$\Lambda(\boldsymbol{k}) = Dk^2 - i\boldsymbol{k}\cdot\boldsymbol{U}. \quad (30)$$

Then it follows from Eq. (12) that the frequency ω in Eq. (20) is replaced by $(\omega-\boldsymbol{k}\cdot\boldsymbol{U})$, and the denominator becomes $D^2k^4+(\omega-\boldsymbol{k}\cdot\boldsymbol{U})^2$. For large wave number $k>k_c$ the higher powers of the k dependence become more important. Therefore, we approximate $\boldsymbol{k}\cdot\boldsymbol{U}=\pm\omega_o$ as a constant in Eq. (25).

III. COMPARISONS WITH PLASMA EXPERIMENTS

During a steady-state plasma discharge, the fluctuations in the magnetized plasma usually have the following properties: the time series of the signals are almost random, as shown by the measured probability density function [10]; the fluctuations are roughly isotropic in the two dimensions transverse to the strong magnetic field [11]; the correlation length of the fluctuations is shorter than the scale lengths of the background density and temperature gradients, which means that the fluctuations are approximately homogeneous.

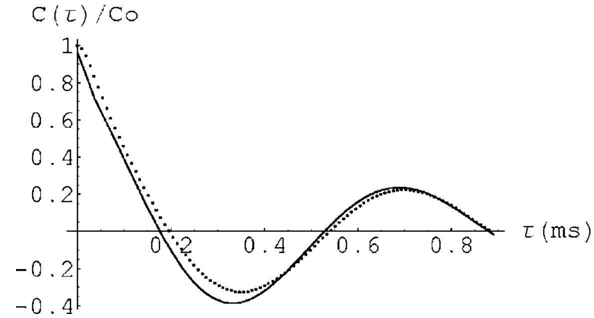

FIG. 2. The dots represent the autocorrelation data of the plasma density fluctuations from helimak experiments. The curve is given by Eq. (25).

Therefore, we may use Eq. (26) to explain the shape and height of the measured auto-power spectrum and Eq. (25) to explain the measured autocorrelation function. When α represents the fractional fluctuations, the mean square value C_o is dimensionless.

As shown in Fig. 1, the theoretical results are compared with the observed plasma density fluctuations from tokamak experiments [9]. The following parameters are used to plot the curve in Fig. 1: $f_o\approx 8$ kHz for the peak frequency, the measured root mean square value $\sqrt{C_o}\approx 0.24$, and $f_c\approx 34$ kHz for the spectrum width. Thus the correlation time $\tau_c=1/f_c\approx 0.03$ ms. For the correlation length $\lambda_\perp\approx 1$ cm, we estimate the local diffusion coefficient $D_\perp=f_c\lambda_\perp^2/(2\pi)^2\approx 0.09$ m^2/s.

The comparison between the theory and the plasma density fluctuations from helimak experiments [12] is shown in Figs. 2 and 3. Note that the autocorrelation data in Fig. 2 and the spectrum data in Fig. 3 were computed from the same time series data. Hence the autocorrelation curve and the spectrum curve are plotted by the same parameters: the observed oscillation frequency $f_o\approx 1420$ Hz, the measured root mean square value $\sqrt{C_o}\approx 0.27$, and $f_c\approx 628$ Hz for the decay rate. Thus the correlation time $\tau_c=1/f_c\approx 1.6$ ms. For correlation length $\lambda_\perp\approx 10$ cm, we estimate the local diffusion coefficient $D_\perp=f_c\lambda_\perp^2/(2\pi)^2\approx 0.16$ m^2/s.

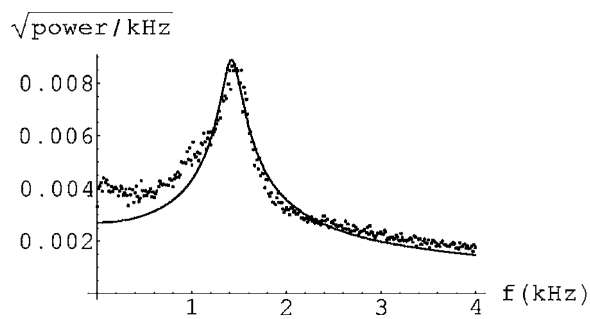

FIG. 3. The dots represent the square root of the autopower spectrum data of the fractional plasma density fluctuations from helimak experiments. The curve is given by the square root of Eq. (26).

Note that in Figs. 1 and 3, the disagreement between the theory and the data at the low and high frequencies (relative to the peak frequency) my be caused by the fact that the measured fluctuations are not perfectly random and isotropic.

In conclusion, we have calculated the autopower spectrum and autocorrelation function of stationary random fluctuations that are homogeneous and isotropic in space. The comparisons show that the results derived from noise theory can explain the autocorrelation function and the autopower spectrum of the observed plasma fluctuations during a steady-state discharge. Our analysis shows that, using the correlation time and the correlation length, we may estimate the transport coefficients of random fluctuations.

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